Cycles, Chords, and Planarity in Graphs

By Damon Hochnadel

A MTH 501 – Mathematical Literature Project

Under the direction of
Prof. John Caughman
With Second Reader
Prof. Paul Latiolais

In partial fulfillment of the requirements for the degree of

Masters of Science in Mathematics
Portland State University
Fariborz Maseeh Department of Mathematics and Statistics

December 3, 2013

I. Introduction

The study of connectivity and planarity of graphs has a rich history that traces back to the early days of graph theory. Much of the early motivation was provided by the desire to prove the famous 4-color theorem, a feat that was finally accomplished in the 1976 by Appel and Haken [3]. Along the way, many related questions arose concerning the structure of planar graphs, and many of these questions continue to generate interesting research. In 1982 Carsten Thomassen conjectured that every longest cycle of a 3-connected graph has a chord [1]. In his article "Every Longest [Cycle] of a 3-Connected, $K_{3,3}$- Minor Free Graph Has a Chord" [1], Etienne Birmelé explores this conjecture for a certain class of 3-connected graphs. The following assessment examines and expands upon Birmelé's arguments.

The title of Bermelé's paper reveals the main goal of the research. Specifically, the aim is to establish the theorem below.

Theorem 1. Every longest cycle of a 3-connected, $K_{3,3}$ - minor free graph has a chord.

To understand this statement and its proof, we must review some of the basic definitions of graph theory. For a more complete introduction, the reader is referred to the fine text by West [2].

II. Definitions

A *graph* G is a triple consisting of a *vertex set* V, an *edge set E*, and a relation that associates with each edge two vertices called its *endpoints.* Two vertices are *adjacent (neighbors)* if they are endpoints of the same edge. A graph is *simple* if it has no loops or multiple edges.

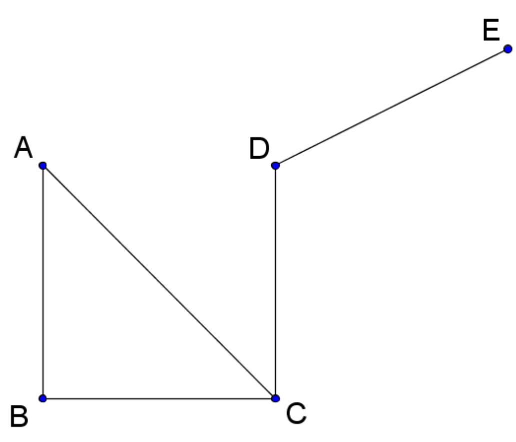


Figure 1 - A simple graph with vertices A, B, C, D, E

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

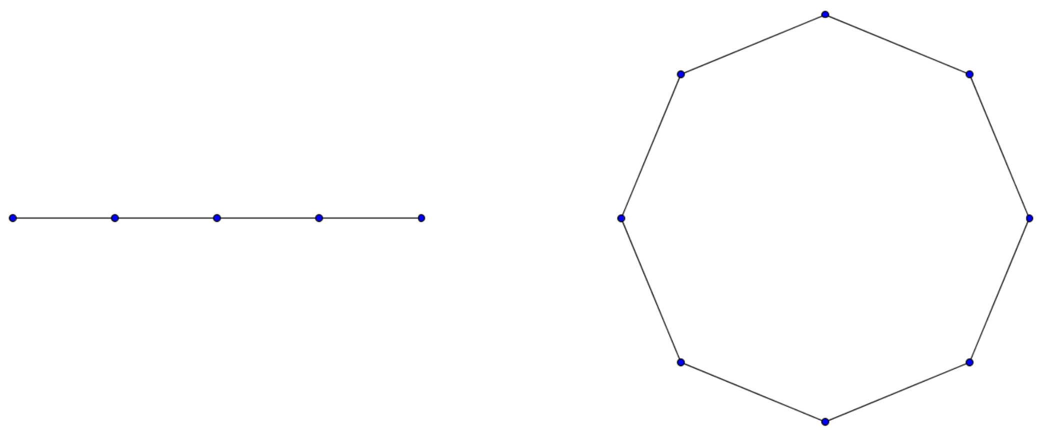


Figure 2 - A path with 5 vertices and a cycle with 8 vertices

A graph G is *connected* if each pair of vertices in G belongs to a path. The *components* of a graph G are its maximal connected subgraphs. The *connectivity* of G is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is *k-connected* if its connectivity is at least k.

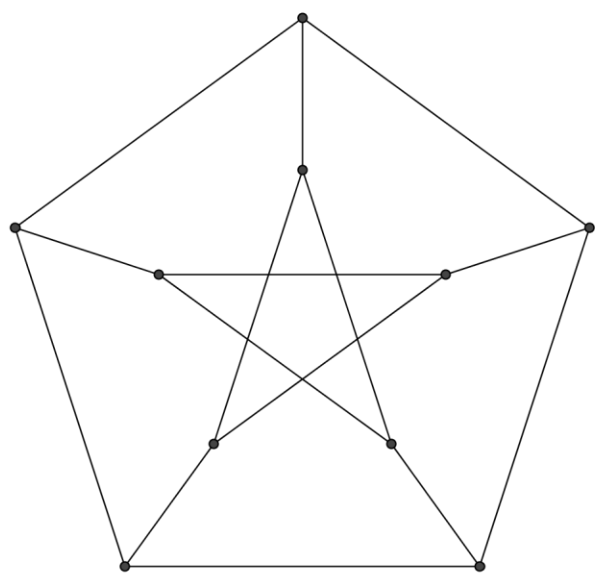


Figure 3 – The Petersen graph is 2-connected but not 3-connected

A *bipartite graph* is a graph where the vertices can be partitioned into two disjoint subsets such that each subset contains no pairwise adjacent vertices. By $K_{n,m}$ we denote the complete bipartite graph where one partite set contains *n* vertices, the other *m*.

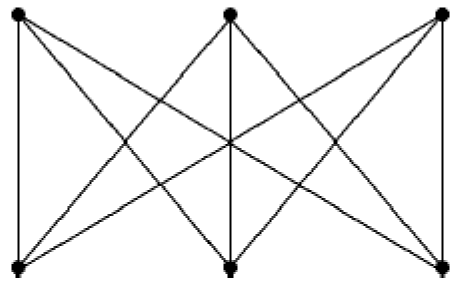


Figure 4 - The Graph $K_{3,3}$

A *minor* of a graph G is a graph H that is obtained from G by a sequence of vertex and edge deletions and edge contractions.

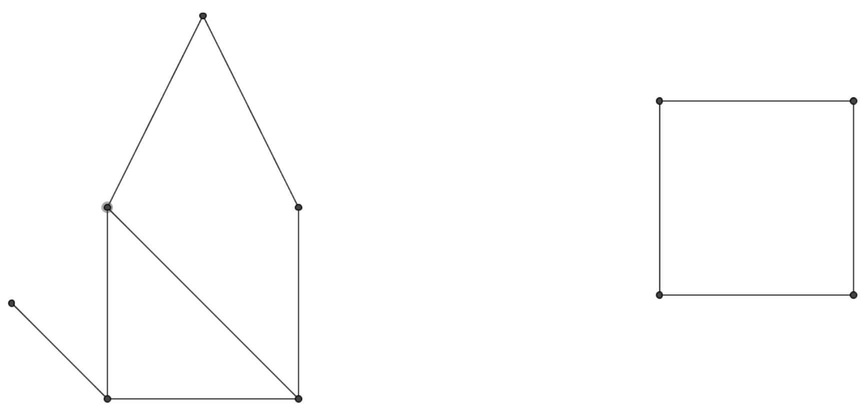


Figure 5 - The graph on the right is a minor of the graph on the left

A cycle $\{v_0, v_1, \ldots, v_n\}$ (we take $v_{n+1} = v_0$) has a *chord* if some v_i is adjacent to v_j where $i \neq j \pm 1$.

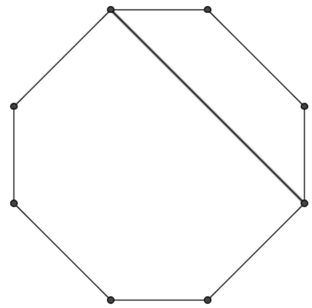


Figure 6 - An eight vertex cycle with a chord.

A graph G is *planar* if it can be represented in the plane without crossing edges. Wagner's Theorem [West, 269] states that a finite graph is planar if and only if it contains no K_5 and no $K_{3,3}$ minors. Therefore, Theorem 1 comes with a corollary.

Corollary 2: Every longest cycle of a 3-connected planar graph has a chord.

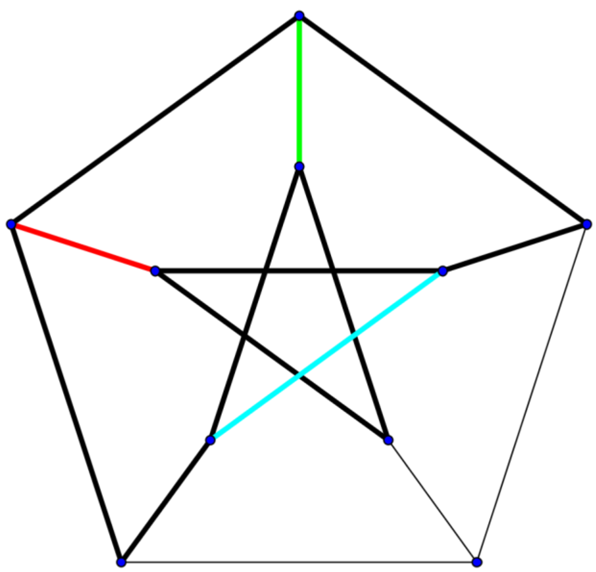


Figure 7 - A longest length cycle of the Petersen graph has multiple chords

III. The Support of a Longest Cycle

We now proceed with our proof of theorem.

Let G be a 3-connected, $K_{3,3}$ - minor free graph. Let C be a cycle of G of longest length. Suppose, by way of contradiction, that C has no chord. Let $v_0, v_1, \ldots, v_p$ be the vertices of C in cyclic order.

Let $H_1, \ldots, H_r$ be the connected components of $G \setminus C$. We denote by $N(i)$ the set of vertices of attachment of the component H_i, that is the set of vertices of C that are adjacent to a vertex of H_i.

Definition 3: Let P be an arc of C, that is a connected subgraph of C, and let $i \in \{1, \ldots, r\}$. We say that P is a *support* of H_i if $N(i) \subseteq V(P)$.

First, we present a short lemma with four observations about these vertices of attachment.

Lemma 4:

(i) $\forall\, i \in \{1, \ldots, r\}$, $|N(i)| \geq 3$.

(ii) $\forall\, k \in \{0, \ldots, p\}$, $\exists\, i \in \{1, \ldots, r\} \ni v_k \in N(i)$.

(iii) Two consecutive integers of C cannot belong to the same $N(i)$.

(iv) There are no integers $(k, l) \in \{0, \ldots, p\}^2$ $(k \neq l)$ and integers $(i, j) \in \{1, \ldots, r\}^2$ such that $\{v_k, v_l\} \subseteq N(i)$ and $\{v_{k+1}, v_{l+1}\} \subseteq N(j)$. (By convention $v_{p+1} = v_0$)

Proof:

(i) As $\forall\, i \in \{1, \ldots, r\}$, the removal of $N(i)$ disconnects H_i from G. Therefore, as G is 3-connected, $|N(i)| \geq 3$.

(ii) If, for some $k \in \{0, \ldots, p\}, k \notin N(i)\ \forall\, i \in \{1, \ldots, r\}$, then all neighbors of v_k lie in C. As C has no chord, v_k would then only have two neighbors, but then v_k could be disconnected from G by the removal of two vertices. Therefore, $\forall\, k \in \{0, \ldots, p\}$, $\exists\, i \in \{1, \ldots, r\} \ni v_k \in N(i)$.

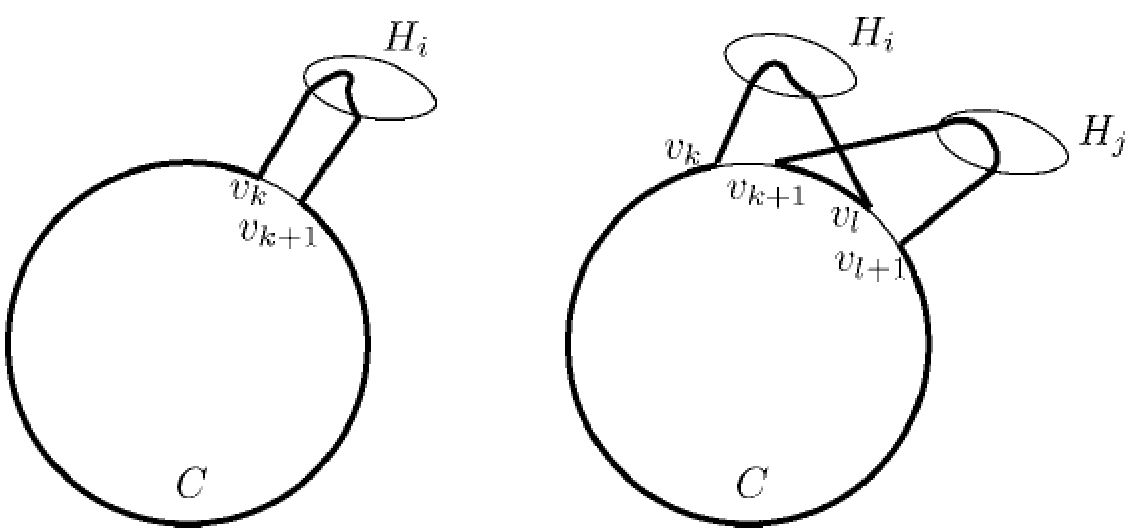


Figure 8 - Lemma 4 part (iii) and (iv)

(iii) and (iv) Figure 8 shows that both cases would contradict the fact that C is the longest cycle in G.

■

IV. Looking At Minimal Supports

Consider a support P for some H_i such that P is minimal with respect to inclusion among all supports. That is, there does not exist a support Q for some component H_j such that Q is a proper subpath of P. We may reorder the vertices of P as $v_0, \dots, v_k$ $2 \leq k \leq p$ (where $|V(C)| = p + 1$, with C being a longest cycle of G). Notice that Pis a proper subpath of C. If $P = C$, then consider $P - e$, where e is any edge of C. As $V(P) = V(P - e)$, $P - e$ is a support for H_i, and $P - e$ is a proper subpath of P, contradicting the minimality of P.

Let P' be the proper subpath of P that shares the same vertex set as P, except for the removal of the two endpoints of P (so P' is the path $v_1, \dots, v_{k-1}$). By Lemma 4, part (iii), $\exists\ \alpha \in V(P') \ni \alpha \notin N(i)$. By Lemma 4 part (ii), $\exists\ j \neq i \in [r] \ni \alpha \in N(j)$.

Let $j \neq i$and Q be a support for H_j such that:

1) $N(j) \cap P'$ is non-empty.
2) $P \cup Q$ is minimal with respect to inclusion. That is, there is no support R for some H_k satisfying the above and such that $P \cup R$ is a proper subpath of $P \cup Q$.
3) Subject to the previous conditions, Q is minimal with respect to inclusion. That is, there is no support R for some H_k such that $N(k) \cap P'$ is non-empty, $P \cup Q = P \cup R$, and R is a proper subpath of Q.

We then choose six (not necessarily totally distinct) vertices, referred to as *vertices of interest*:

-Three distinct vertices of N(i) (must be on P, as P is a support of H_i), including the two endpoints of P.

-Three distinct vertices of N(j) (must be on Q, as Q is a support of H_j), including both endpoints of Q, and one on P' (note that the vertex on P' may be an endpoint of Q).

So, we have at least three vertices of interest, and at most six.

First, we demonstrate that $P \cup Q$ is a proper subpath of C.

Suppose that the above held and that $P \cup Q = C$. Then, as $P \subset C$, there exists an edge e of C such that $P \subseteq C - e$. We can then rename $C - e$ as Q'. Note that, like Q, Q' is a support for H_j, and $P \cup Q'$ is a proper subpath of $P \cup Q$, contradicting the minimality of $P \cup Q$.

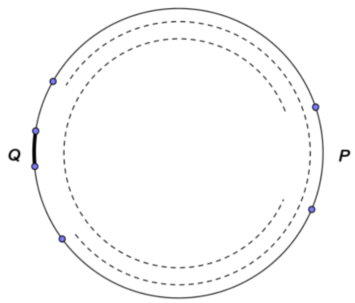


Figure 9 - Cycle Diagram 1

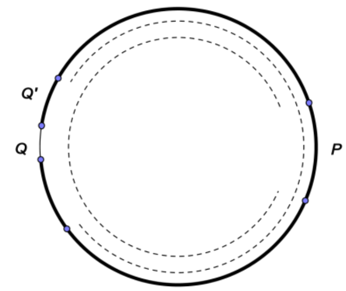


Figure 10 - Cycle Diagram 2

We now separate the rest of our proof into twelve cases. Notice that in all figures, the bold arcs may not be more than a single vertex, unless stated otherwise:

V. Finding a $K_{3,3}$

Case 1: P is a subpath of Q.

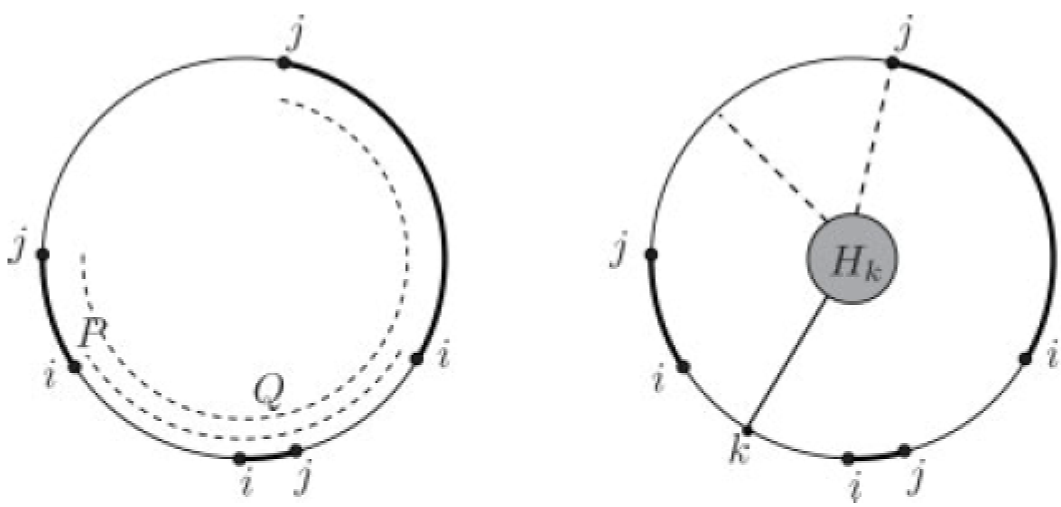


Figure 11 - The case when P is a subset of Q

In examining the diagram, recall that the bold arc to the left (possibly a single vertex) has one vertex of N(i) that is an endpoint of P, and one vertex of N(j) that is an endpoint of Q (similarly with the right arc). The center bold arc (possibly a single vertex) has one vertex of N(i) taken from somewhere on P′ (by Lemma 4 part (i) and the fact that P is a support of H_i, N(i) ∩ P′ is non-empty) as well as one vertex of N(j) taken from somewhere on P′.

Claim: ∃ k ∈ [r] ∋ i ≠ k ≠ jand that some vertex of N(k) lies on a sub-arc of P that is not bold.

Proof: Suppose there is no vertex of N(k) (for any k ∈ [r]) in the non-bold sub-arc of P to the left (on the diagram). Then, by assumption and Lemma 4 part (iii), there must be one and only one vertex within this sub-arc of P, and it must be in N(j). Now, consider the non-bold sub-arc of P to the right. The first vertex (proceeding counterclockwise) after the right endpoint of the central bold sub-arc can not belong to N(j) (else it would violate Lemma 4 part (iii)), and it can not belong to N(i) (else it would violate Lemma 4 part (iv), using the single vertex in N(j) found in the other non-bold sub-arc of P followed by the vertex on the center bold arc in N(i)), therefore this adjacent vertex must belong to some N(k), where i ≠ k ≠ j.■

If a vertex of some N(k), i ≠ k ≠ jlies in the non-bold sub-arc of P on the left, then we have the right diagram of Figure 10. Otherwise, there is exactly one vertex on this sub-arc and it is in N(j), and we have a vertex in N(k) in the non-bold sub-arc to the right. We then replace the j end of the central bold sub-arc of P with the vertex in the left non-bold sub-arc, giving us a new central bold arc and a case that is the same as that in Figure 11 by reflecting the image across a vertical line.

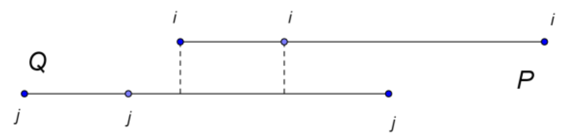
Claim: By the minimality of P ∪ Q and Q we must either have both endpoints of P ∪ Q in N(k), or there must be some vertex of V(C)\V(P ∪ Q) in N(k).

Proof: Suppose no vertices of N(k) lie in V(C)\V(P ∪ Q) and at least one endpoint of P ∪ Qis not in N(k).
Then, let R be a subarc of Q such that V(R) = V(Q) − v where vis one endpoint of Q such that v ∉ N(k).Then, as v ∉ N(k)and there are no vertices of N(k) on V(C)\V(P ∪ R), R includes all vertices of N(k), and therefore R is a support of H_k. Notice that from our prior claim proof, N(k) ∩ P′ is non-empty. Also, R is a proper subpath of Q. Therefore, P ∪ R is a proper subpath of P ∪ Q or P ∪ R = P ∪ Q and R is a proper subpath of Q, contradicting the minimality of our selection of Q. ■

Therefore, one of the two dotted lines in Figure 11 exists. We then let the three bold arcs of C be $A_1, A_2, A_3$and we let $B_1 = H_i, B_2 = H_j,$and $B_3 = H_k \cup (V(C)\backslash V(A_1 \cup A_2 \cup A_3) \cap N(k))$. These six subsets of V(G) are disjoint and connected and each A_i is linked to each B_j by at least one edge. Therefore, by contracting each A_i and B_j to a single vertex, we obtain a $K_{3,3}$ minor of G.

Cases two through twelve are the elven distinct up to symmetry configurations when Pis not a subpath of Q.

Case 2: Pis not a subpath of Q, and the vertices of interest are ordered along P ∪ Q as indicated.

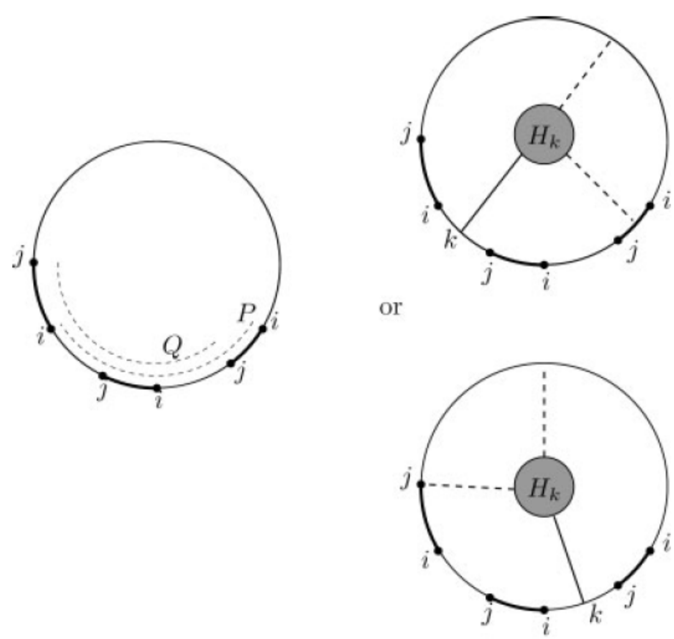


Case 2

By Lemma 4 part (iii), there must be some additional vertex between the left-most vertices of N(i).

First, suppose that the only vertices there were of N(j), then we may change our vertices of interest, replacing one from N(j) with a new one in the middle. We then produce the situation shown in Figure 12 (Case 2(a)).

Figure 12 - Case 2(a)



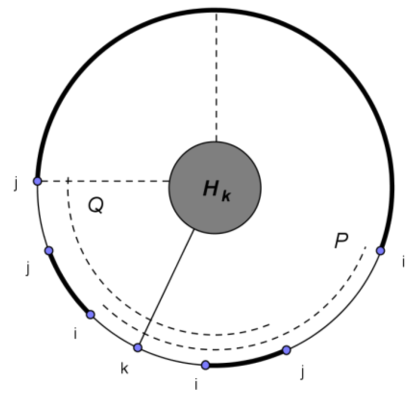
We then have two sub-cases for this case. There are two non-bold sub-arcs of P ∪ Q. As each one has an i vertex of interest on one side and a j vertex of interest on the other, at least one of the two non-bold sub-arcs must have a vertex on it belonging to some N(k), where i ≠ k ≠ j,or else we would violate Lemma 4 part (iv). If we have such a vertex in the left non-bold sub-arc then we have the first case on the right of Figure 12, else we have the second case on the right of Figure 12.

In the first sub-case suppose there are no vertices of $N(k)$ in the right-most bold sub-arc, and there are none on $V(C)\backslash V(P \cup Q)$,then letting R be the subarc of Q such that $V(R) = V(Q) - v$ where v is the right endpoint of Q. As $v \notin N(k)$and there are no vertices of $N(k)$ on $V(C)\backslash V(P \cup Q)$, R includes all vertices of $N(k)$, and is therefore a support of H_k. Note that as the k vertex indicated exists, $N(k) \cap P'$ is non-empty. Also, R is a proper subpath of Q. Therefore, $P \cup R = P \cup Q$ (as v lies on P) and R is a proper subpath of Q, contradicting the minimality of our selection of Q. Therefore, one of the two dotted lines must be an edge. We then produce a $K_{3,3}$minor as before with the three bold arcs being A_1, A_2, A_3 and $B_1 = H_i, B_2 = H_j, B_3 = H_k \cup (V(C)\backslash V(A_1 \cup A_2 \cup A_3) \cap N(k))$. Each of the A_i and B_j are disjoint, connected, and each $A_i$is linked to each $B_j$by at least one edge, obtaining the $K_{3,3}$ - minor.

In the second sub-case suppose there are no vertices of $N(k)$ on $V(C)\backslash V(P \cup Q)$ and the left endpoint of Q is not in $N(k)$, then we let R be the subarc of $P \cup Q$ such that $V(R) = V(P \cup Q) - v$ where v is the left endpoint of Q. As $v \notin N(k)$and there are no vertices of $N(k)$ on $V(C)\backslash V(P \cup Q)$, R includes all vertices of $N(k)$, and is therefore a support of H_k. Note that as the k vertex indicated exists, $N(k) \cap P'$ is non-empty. If $v \in V(P)$, then $P \cup Q = P$, so R, which is a proper subpath of $P \cup Q$ is a proper subpath of P,contradicting the minimality of the selection of P. Otherwise, as $V(R) = V(P \cup Q) - v$, and $v \in V(Q)$, but $v \notin V(P)$, $P \cup R$is a proper subpath of $P \cup Q$, contradicting the minimality of our selection of Q. Therefore, one of the two dashed lines is an edge. We then produce a $K_{3,3}$minor as before with the three bold arcs as $A_1, A_2,$and $A_3$and $B_1 = H_i, B_2 = H_j,$and $B_3 = H_k \cup (V(C)\backslash V(A_1 \cup A_2 \cup A_3) \cap N(k))$. Each are disjoint and connected and each $A_i$is linked to each $B_j$by at least one edge, obtaining the $K_{3,3}$ - minor.

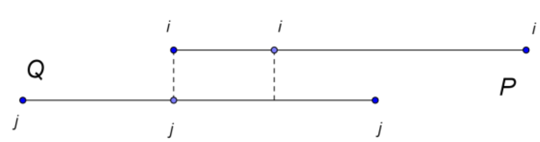
Suppose, instead, there were a vertex u of some $N(k)$, $(i \neq k \neq j)$between the two left-most vertices of interest of $N(i)$ as in Figure 13 (Case 2(b)).

Figure 13 - Case 2(b)



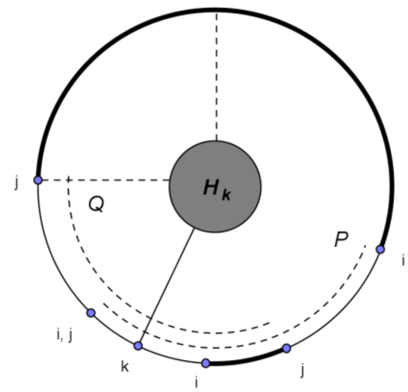
Let the endpoint of Q not lying on Pbe v, and suppose, by way of contradiction, that $(V(C)\backslash V(P \cup Q - v)) \cap N(k)$ is the empty set. Then, $P \cup Q - v$ is a support for $N(k)$, and by the placement of u we have $N(k) \cap P'$ is non-empty, contradicting our selection of Q, as $P \cup Q - v$ is a proper subpath of $P \cup Q$. Therefore, there exists some vertex of $N(k)$ on $V(C)\backslash V(P \cup Q - v)$. As before, we then produce a $K_{3,3}$ minor using the three bold subarcs as A_1, A_2, A_3 and $B_1 = H_i, B_2 = H_j,$and $B_3 = H_k \cup ((V(C)\backslash V(A_1 \cup A_2 \cup A_3)) \cap N(k))$.

Case 3: Figure 14



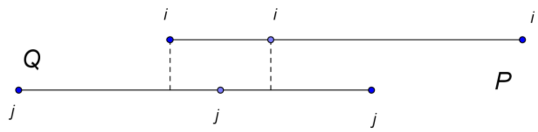
Case 3

Figure 14



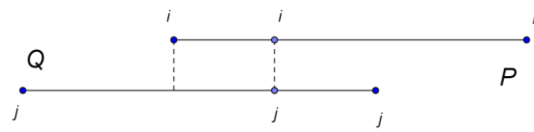
Notice that were we to contract the bold arc lying on Q in case 2(b) at the beginning of that case, we would have case 3. Also, the only part that must be changed from case 2(b) is that first possibility of only vertices from N(j) lying between the two vertices of N(i)is not possible, as one of the vertices of N(i) also is in N(j), we must therefore have a vertex beloning to some N(k), $(i \neq k \neq j)$.Otherwise, this case proceeds as case 2(b), as those two vertices being distinct was not necessary to prove case 2(b).

Case 4: This case appeared as a possibility during case 2(a), and thus has already been shown.



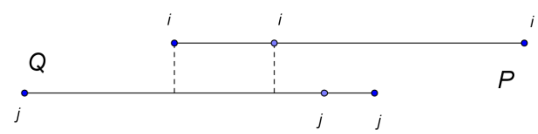
Case 4

Case 5: Had we contracted the bold arc of case 4 that lay on P ∩ Q, we would produce case 5. As these vertices being distinct was not necessary, this case has already been shown.



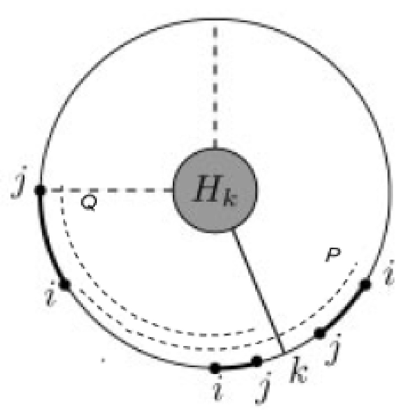
Case 5

Case 6: Figure 15



Case 6

Figure 15

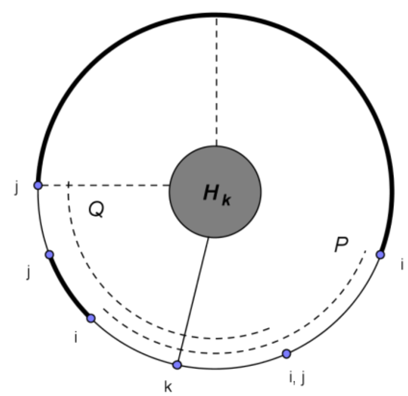


By Lemma 4 part (iii), there must be at least one vertex from some N(k) for some k ≠ jbetween the two vertices of interest from N(j) to the right. Suppose there are only vertices of N(i), then were we to replace the central bold subarc of P ∪ Q with one from a vertex of N(i) between the two bold subarcs with the central vertex of interest of N(j) we find that this is as in case 3. Therefore, we may assume there is a vertex of N(k) for some i ≠ k ≠ jbetween the two vertices of interest from N(j) on the right. At this point our proof proceeds in the same manner as the second sub-case of case 2(a), giving us a $K_{3,3}$ - minor.

Case 7: Figure 16

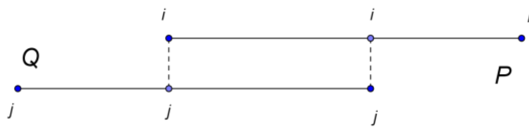
Case 7

Figure 16



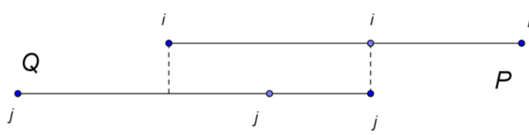
As the end vertices in the right bold subarc of P ∪ Q in case 2 were not required to be distinct, this case proceeds as in case 2.

Case 8: Like case 7, this proceeds as a special case of case 2.



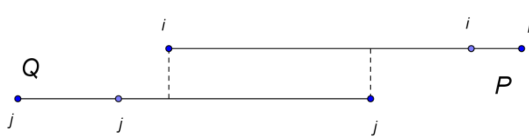
Case 8

Case 9: This is a special case of case 4.



Case 9

Case 10:



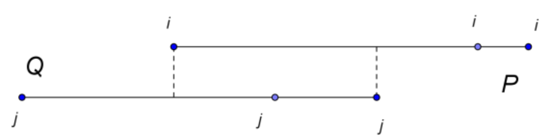
Case 10

Case 10 proves to be a problem case. It does not allow for the same process as the other cases as we can not use a similar argument to force a vertex of some H_k upon $Q \cap P$. There is also no alternative argument readily apparent to this author, nor found within Birmelé's article to either exclude or deal with it. Therefore, we leave this problem case unresolved.

Case 11: This case will be completed as a special case of case 12.

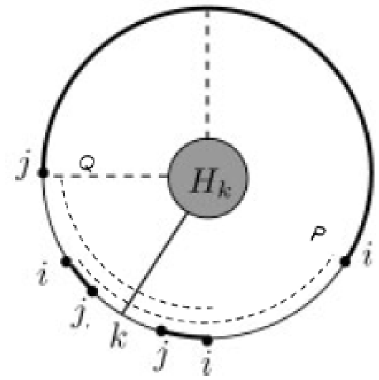
Case 11

Case 12: Figure 17



Case 12

Figure 17



By Lemma 4 part (iii), there must be at least one vertex from some N(k) for some $k \neq j$between the two vertices of interest from N(j) to the right. Suppose there are only vertices of N(i), then the left of the two bold arcs on $P \cup Q$ must not be a single vertex (by Lemma 4 part (iii)), and this becomes case 4. Therefore, we may assume there is a vertex of N(k) for some $i \neq k \neq j$between the two vertices from N(j) on the right. Suppose that there are no vertices of N(k) on $V(C)\backslash V(P \cup Q)$ and the left endpoint of Q is not in N(k), then we let R be the subarc of $P \cup Q$ such that $V(R) = V(P \cup Q) - v$ where v is the left endpoint of Q. As $v \notin N(k)$and $N(k) \cap (V(C)\backslash V(P \cup Q))$ is empty, R is a support of H_k. As the left endpoints of Q and P are distinct, $v \notin V(P)$.Therefore, as $V(R) = V(P \cup Q) - v$ and $v \in V(Q)$, but $v \notin V(P), P \cup R$is a proper subpath of $P \cup Q$, violating the minimality of the selection of Q. Therefore, either $v \in N(k)$ or $N(k) \cap (V(C)\backslash V(P \cup Q))$ is non-empty. Therefore, one of the two dashed lines is an edge. We then produce a $K_{3,3}$ minor as before with the three bold arcs as $A_1, A_2,$and $A_3$and $B_1 = H_i, B_2 = H_j,$and $B_3 = H_k \cup (V(C)\backslash V(A_1 \cup A_2 \cup A_3) \cap N(k))$. Each are disjoint and connected and each $A_i$is linked to each $B_j$by at least one edge, obtaining the $K_{3,3}$ minor. ■

VI. Conclusion

Throughout this paper, we have endeavored to prove that every the longest cycle of any 3-connected, $K_{3,3}$ - minor free graph must contain a chord using the techniques argued in Birmelé's article [1] on the subject. Using the 3-connectedness of the graph together with examining the supports of the components of $G \setminus C$, we manage to see that assuming the lack of a chord forces the graph to contain a $K_{3,3}$ - minor in all but one case. With further research into the instance of this case, it could be possible to either have a complete proof of a large collection of graphs covered in Thomassen's conjecture on the matter, or a possible counterexample to the conjecture.

References

1. Etienne Birmelé, Every Longest Circuit of a 3-Connected, $K_{3,3}$ - Minor Free Graph Has a Chord, J Graph Theory 58 (2008), 293-298.

2. Douglas B. West, Introduction to Graph Theory, Second Edition, Prentice-Hall, Inc., Upper Saddle River, NJ, 1996.

3. Four Color Theorem, Wikipedia, http://en.wikipedia.org/wiki/4_color_theorem.